# Approximate controllability of Boussinesq channel flows only through the temperature

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**Abstract**

We show the global approximate controllability of the Boussinesq system with viscosity and diffusion in a planar periodic channel by using only a temperature control supported in a thin strip. At the walls, a slip boundary condition is imposed on the fluid and the normal derivative of the temperature is assumed to vanish. This contributes a first global controllability result of such type for the Boussinesq system in the presence of physical boundaries. For the proof, we resort to a small-time scaling argument to control the vorticity through a large initial temperature. Moreover, J.-M. Coron's return method is employed in order to steer the temperature without significantly impacting the vorticity.

## 1 Introduction

The global (large data) approximate controllability of the Boussinesq system by using solely a localized temperature control was recently tackled in [29] when the domain is the 2D flat torus. This demonstrates for the fully periodic setting that, given any initial configuration, the system can approximately reach any target in arbitrary time if one acts with an appropriate physically localized force on the temperature. However, for domains with boundaries the question of global controllability only through the temperature has remained unanswered. The present article provides now the existence of at least one setting with boundaries for which global controllability properties can be achieved by using merely a localized temperature control. More specifically, we prove the global approximate

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controllability of the Boussinesq system in a straight thermally insulated periodic channel with a slip boundary condition for the fluid, and driven by an interior temperature control supported in a strip of nonzero width. The main challenges of the considered controllability problem are as follows: 1) the controls can only act on the temperature equation and should be physically localized; 2) initial and target states can be far away in the state space; 3) the temperature enters the momentum equation only in the direction of gravity. Our approach rests on a scaling argument to control the vorticity through the initial temperature. Furthermore, we employ the return method (see [8, Part 2, Chapter 6]) for steering the temperature. From the viewpoint of applications, the Boussinesq system is relevant, *e.g.*, to the study of geophysical phenomena, Rayleigh-Bénard convection, heating, and ventilation (see [1, 6, 16, 19, 24]).

We consider viscous incompressible flows under Boussinesq heat effects in the channel $\mathscr{C} := (-1, 1) \times \mathbb{T}$, where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. The fluid is assumed to slip without friction along the solid boundary $\Gamma := \partial\mathscr{C} = \{-1, 1\} \times \mathbb{T}$, while for the temperature we prescribe the zero Neumann boundary condition at $\Gamma$. To influence the dynamics, only external heating/cooling, realized by an interior control $\eta \colon \omega \times (0, T) \longrightarrow \mathbb{R}$, can be applied in a control zone of the form

$$\omega := [-1, 1] \times [a, b], \quad 0 \leq a < b < 2\pi.$$

The velocity $\boldsymbol{u} \colon \mathscr{C} \times [0, T] \longrightarrow \mathbb{R}^2$, temperature $\theta \colon \mathscr{C} \times [0, T] \longrightarrow \mathbb{R}$, and exerted pressure $p \colon \mathscr{C} \times [0, T] \longrightarrow \mathbb{R}$ are governed by the controlled Boussinesq system

$$\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} + \boldsymbol{\nabla} p = \theta \boldsymbol{g} + \boldsymbol{\Phi}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$
$$\partial_t \theta - \tau \Delta \theta + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\theta = \mathbb{I}_\omega \eta + \psi, \tag{1.1}$$
$$\boldsymbol{u}|_\Gamma \cdot \boldsymbol{n} = 0, \quad (\boldsymbol{\nabla} \wedge \boldsymbol{u})|_\Gamma = 0, \quad \partial_{\boldsymbol{n}} \theta|_\Gamma = 0, \quad \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \quad \theta(\cdot, 0) = \theta_0$$

where $\boldsymbol{n}$ is the outward unit normal at $\Gamma$, the constant vector $\boldsymbol{g} := [0, 1]^\top$ refers to gravity, $(\boldsymbol{u}_0, \theta_0)$ are the initial states, and $(\boldsymbol{\Phi}, \psi)$ are given external forces (*e.g.*, $\boldsymbol{\Phi} = \boldsymbol{0}$ and $\psi = 0$). The evolution equations in (1.1) consist of a Navier–Stokes system for the fluid, where temperature enters as a force in the direction of gravity, and of a convection diffusion equation, with the fluid velocity as drift field, for the temperature. As discussed in Section 2, the problem (1.1) is globally wellposed for the considered classes of data. The boundary condition for $\boldsymbol{u}$, which allows the fluid to slip at the flat boundary $\Gamma$, is sometimes called the Lions free boundary condition (see [23, Page 129], [33, Remark 2.4], and [11, Section 1.5.3]). It can also be seen as a limiting case of the Navier slip-with-friction boundary condition dating back to [26]. As made precise below in Theorem 1.1, the approximate controllability of (1.1) in the present context essentially means that, for any $\varepsilon > 0$, initial configuration $(\boldsymbol{u}_0, \theta_0)$, target $(\boldsymbol{u}_T, \theta_T)$, and suitable forces $(\boldsymbol{\Phi}, \psi)$, one can choose the control $\eta$ in (1.1) such that $\|(\boldsymbol{u}, \theta)(\cdot, T) - (\boldsymbol{u}_T, \theta_T)\|_X < \varepsilon$, where X will be a product space of Sobolev type.

When controllability properties are known for a system, it is natural to ask whether, or to which extend, such properties remain true if one limits the actions of

the controls to a reduced number of components. For the Navier–Stokes equations with the no-slip boundary condition, the local null controllability with controls having two vanishing components has been shown in [10]. Further, see [9] (periodic boundary conditions) and [21] (Navier type boundary conditions) for local exact controllability results with controls having one vanishing component. However, there are currently no such global controllability results for the Navier–Stokes system, also not if slip or periodic boundary conditions are employed. And even when allowing physically localized controls to act directly in all components of the system, the global approximate controllability of the 2D and 3D Navier–Stokes equations with the no-slip boundary condition is a well-known open problem posed by J.-L. Lions (see [11, 12] and the references therein). Also the global stabilization of the Navier–Stokes equations presents challenges; we refer to [13, 35] for recent developments and further literature. Concerning controls that are localized in frequency, we refer to [2, 31] and to [28] for recent progress and literature remarks. In particular, several types of global approximate controllability, including trajectorial and Lagrangian controllability, with finite-dimensional controls vanishing in two components, have been obtained in [27] for the Navier–Stokes system on the 3D flat torus. Physically localized and highly degenerate controls acting through all components of the Navier–Stokes system on the 2D flat torus have been studied recently in [30].

Concerning the Boussinesq system driven only by a physically localized temperature control, several local exact controllability results are due to [14]. An argument requiring less assumptions on the domain compared with [14] is given in [4], and cases with Navier type boundary conditions have been considered in [25]. Local exact controllability of the latter type could be combined with our main theorem in order to conclude results on the global exact controllability to the zero state, or to suitable trajectories, by using only a physically localized temperature control. When the controls can act in the equations for the velocity and the temperature, local exact controllability has been studied, *e.g.*, in [17, 18, 20], and global controllability results relying on the return method are provided in [5, 15, 17].

## 1.1 Main result

Let $\overline{\mathrm{H}}$ consist of all $f \in \mathrm{L}^2(\mathscr{C}; \mathbb{R})$ with $\int_{\mathscr{C}} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0$ and write $\boldsymbol{\nabla} \wedge \boldsymbol{f} = \partial_1 f_2 - \partial_2 f_1$ for the "curl" of $\boldsymbol{f} = [f_1, f_2]^\top$. The notation $\|\cdot\|$ refers either to $\|\cdot\|_{\mathrm{L}^2(\mathscr{C};\mathbb{R})}$ or to $\|\cdot\|_{\mathrm{L}^2(\mathscr{C};\mathbb{R}^2)}$. Moreover, we define the spaces

$$\mathbf{H} := \left\{ \boldsymbol{f} \in \mathrm{L}^2(\mathscr{C}; \mathbb{R}^2) \ \middle| \ \boldsymbol{\nabla} \cdot \boldsymbol{f} = 0 \text{ in } \mathscr{C}, \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ at } \Gamma, \boldsymbol{f} \cdot \boldsymbol{g} \in \overline{\mathrm{H}} \right\},$$

$$\mathrm{H}_0^m := \mathrm{H}^m(\mathscr{C}; \mathbb{R}) \cap \mathrm{H}_0^1(\mathscr{C}; \mathbb{R}), \quad m \geq 1,$$

$$\mathrm{H}_{\mathrm{N}}^m := \left\{ f \in \mathrm{H}^m(\mathscr{C}; \mathbb{R}) \cap \overline{\mathrm{H}} \ \middle| \ \partial_{\boldsymbol{n}} f = 0 \text{ at } \Gamma \right\}, \quad m \geq 2,$$

endowed respectively with $\| \cdot \|$ and $\| \cdot \|_m := (\sum_{|\alpha| \leq m} \|\partial^\alpha \cdot \|^2)^{1/2}$. Let us also recall the elliptic regularity estimate for div-curl problems (see [34, Apendix I])

$$\|f\|_k \lesssim \|\nabla \cdot f\|_{k-1} + \|\nabla \wedge f\|_{k-1} + \|f \cdot n\|_{\mathrm{H}^{k-1/2}(\Gamma;\mathbb{R})} + |\int_{\mathscr{C}} f(x) \cdot g \, dx|, \quad (1.2)$$

where $a \lesssim b$ means $a \leq Cb$ and $C > 0$ shall always refer to an absolute constant. The last term on the right-hand side in (1.2) appears since $\mathscr{C}$ is doubly-connected and its de Rham first cohomology space is spanned by $g = [0,1]^\top$, which obeys $\nabla \wedge g = 0$, $\nabla \cdot g = 0$, and $g|_\Gamma \cdot n = 0$. Throughout, we assume that the Lebesgue measure is normalized so that $\int_{\mathscr{C}} dx = 1$.

**Theorem 1.1** (Main result). *The system* (1.1) *is globally approximately controllable by using only a temperature control. That is, for arbitrary viscosity $\nu > 0$, diffusivity $\tau > 0$, control time $T > 0$, accuracy $\varepsilon > 0$, states $(u_0, \theta_0) \in \mathbf{H} \times \overline{\mathrm{H}}$, $(u_T, \theta_T) \in \mathbf{H} \times \mathrm{H}_{\mathrm{N}}^2$ with $\nabla \wedge u_T \in \mathrm{H}_0^1$, and forces $(\Phi, \psi) \in \mathrm{H}^1((0,T); \mathbf{H} \times \overline{\mathrm{H}}) \cap \mathrm{L}^2((0,T); \mathrm{H}^1(\mathscr{C}; \mathbb{R}^{2+1}))$, there exists a control $\eta \in \mathrm{C}^\infty(\mathscr{C} \times [0,T]; \mathbb{R})$ with $\mathrm{supp}(\eta) \subset \omega \times (0,T)$ such that the unique solution $(u, \theta)$ to* (1.1) *satisfies*

$$\|u(\cdot, T) - u_T\|_2 + \|\theta(\cdot, T) - \theta_T\|_2 < \varepsilon.$$

**Remark 1.2.** Without any further assumptions, controlled solutions $(u, \theta)$ obtained via Theorem 1.1 are of Leray-Hopf type (see also [5]) and belong to the space

$$C^0([0,T]; \mathbf{H} \times \mathrm{L}^2(\mathscr{C}; \mathbb{R})) \cap \mathrm{L}^2((0,T); \mathrm{H}^1(\mathscr{C}; \mathbb{R}^{2+1})).$$

Accounting for classical parabolic smoothing effects (*e.g.*, see [5,34]), one further has $(u, \theta) \in C^0([t_0, T]; \mathrm{H}^2(\mathscr{C}; \mathbb{R}^{2+1}))$ for any $t_0 > 0$. Following the lines of [32, Theorem 3.1], if $(u_0, \theta_0) \in \mathbf{H} \times \mathrm{H}_{\mathrm{N}}^2$ with $\nabla \wedge u_0 \in \mathrm{H}_0^1$ in Theorem 1.1, one can show that $(u, \theta) \in C^0([0,T]; \mathrm{H}^2(\mathscr{C}; \mathbb{R}^{2+1}))$.

**Remark 1.3.** As detailed in [11, Section 2.6 and Section 5] (see also [5,22]), one could combine Theorem 1.1 with local exact controllability results to obtain global exact ones. We refer to [4,14,25] for local results that could be adapted for this purpose. Moreover, by density, Theorem 1.1 likewise provides approximate controllability in $\mathrm{L}^2 \times \mathrm{L}^2$ or $\mathrm{H}^1 \times \mathrm{H}^1$, and also higher order Sobolev spaces could be considered with the present method.

**Remark 1.4.** Regarding boundary conditions modeling friction at the walls, difficulties would arise in the context of Theorem 2.6, where viscous boundary layers would enter the proof. For Navier slip-with-friction boundary conditions, it was shown in [5,11] how these boundary layers can be dissipated sufficiently by using velocity controls. For the no-slip boundary condition, and even with controls in both the velocity and the temperature equations, the question of global approximate controllability parallels the open problem due to J.-L. Lions mentioned above. Also for the application of Theorem 2.1 in Section 2.4, we rely on the imposed zero Dirichlet trace of the vorticity. The shape of the domain and the type of the control regions considered here facilitate moreover a suitable return method trajectory for (1.1) controlled only through the temperature (see Remark 2.3).

## 2 Proof of the main result

The proof of Theorem 1.1 involves two key ingredients: approximate controllability of the vorticity in a short time via a large control acting through the initial temperature (see Theorem 2.1 in Section 2.2); a return method argument combined with a localization strategy (Section 2.3). Finally, Theorem 1.1 is concluded in Section 2.4.

### 2.1 Remarks on the Boussinesq system

As the Boussinesq system (1.1) accounts for viscosity and heat diffusion, its global wellposedness and regularity theories parallel those for the 2D Navier–Stokes system; see, *e.g.*, [16,33,34], and also [5] for controlled Leray-Hopf weak solutions.

The vorticity formulation of the system (1.1), posed in $\mathbb{T}^2 \times [0,T]$ with $T > 0$, is obtained by formally acting with "$\nabla \wedge$" on the velocity equation:

$$\partial_t w - \nu \Delta w + (\boldsymbol{u} \cdot \nabla)w = \partial_1 \theta + \varphi,$$

$$\partial_t \theta - \tau \Delta \theta + (\boldsymbol{u} \cdot \nabla)\theta = \mathbb{I}_\omega \eta + \psi,$$

$$\nabla \wedge \boldsymbol{u} = w, \quad \nabla \cdot \boldsymbol{u} = 0, \quad \int_{\mathscr{C}} \boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{g} \, d\boldsymbol{x} = \int_0^t \int_{\mathscr{C}} \theta(\boldsymbol{x},s) \, d\boldsymbol{x} ds, \quad (2.1)$$

$$\boldsymbol{u}|_\Gamma \cdot \boldsymbol{n} = 0, \quad (\nabla \wedge \boldsymbol{u})|_\Gamma = 0, \quad \partial_{\boldsymbol{n}} \theta|_\Gamma = 0,$$

$$w(\cdot,0) = w_0, \quad \theta(\cdot,0) = \theta_0,$$

where the functions $w_0 = \nabla \wedge \boldsymbol{u}_0$, $\varphi = \nabla \wedge \boldsymbol{\Phi}$, and $\psi$ are those prescribed in Theorem 1.1. If $\mathbb{I}_\omega \eta(\cdot,t)$ is average-free for $t \in [0,T]$, then also $\theta(\cdot,t)$. Moreover, due to (1.2), the average of $\boldsymbol{u}(\cdot,t) \cdot \boldsymbol{g}$ must be specified for all $t \in [0,T]$ to ensure uniqueness of solutions on $[0,T]$. Further, assuming $\mathbb{I}_\omega \eta$ to be sufficiently regular and $(\boldsymbol{u}_0, \theta_0) \in \mathbf{H} \times \mathrm{H}_N^2$ with $\nabla \wedge \boldsymbol{u}_0 \in \mathrm{H}_0^1$, we denote by

$$S(w_0, \theta_0, \varphi, \mathbb{I}_\omega \eta + \psi) := (w, \theta) \in \mathrm{C}^0([0,T]; \mathrm{H}_0^1 \times \mathrm{H}_N^2) \cap \mathrm{L}^2((0,T); \mathrm{H}_0^2 \times \mathrm{H}_N^3)$$

the respective solution to (2.1), and by $S_t = (S_t^1, S_t^2)$ its restriction at time $t$; *i.e.*,

$$S_t^1(w_0, \theta_0, \varphi, \mathbb{I}_\omega \eta + \psi) := w(\cdot,t), \quad S_t^2(w_0, \theta_0, \varphi, \mathbb{I}_\omega \eta + \psi) := \theta(\cdot,t). \quad (2.2)$$

In Theorem 1.1, it is not necessary to assume $\nabla \wedge \boldsymbol{u}_0 \in \mathrm{H}_0^1$ and $\theta_0 \in \mathrm{H}_N^2$, as parabolic smoothing effects will regularize the velocity and temperature in any short time; see "Step 1" in Section 2.4.

Further, let us recall that for given $w \in \mathrm{H}^m(\mathscr{C}; \mathbb{R})$, $m \in \mathbb{N} \cup \{0\}$, and $A \in \mathbb{R}$ the div-curl problem

$$\nabla \wedge \boldsymbol{u} = w, \quad \nabla \cdot \boldsymbol{u}, \quad \int_{\mathscr{C}} \boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{g} \, d\boldsymbol{x} = A$$

has the unique solution $\boldsymbol{u} := \nabla^\perp \zeta + A\boldsymbol{g} \in \mathrm{H}^{m+1}(\mathscr{C}; \mathbb{R}^2)$, where $\zeta$ solves $\Delta \zeta = -w$ in $\mathscr{C}$ with the boundary condition $\zeta|_\Gamma = 0$.

## 2.2 Steering the vorticity through the initial temperature

The next theorem allows to control the vorticity in a short time by means of a large initial temperature. Such an argument was introduced in [3] for the primitive equations in $\mathbb{T}^3$ and in [29] for the Boussinesq system in $\mathbb{T}^2$. Here, we develop the respective argument from [29] for the present setting with boundary conditions at $\partial\mathscr{C}$. Due to the boundary effects, we can only reach approximately the vorticity states of the form $w_0 - \partial_1 \xi$ for sufficiently regular $\xi$ which satisfy $\partial_1 \xi|_\Gamma = 0$ and the technical condition $\partial_{111}\xi|_\Gamma = 0$.

**Theorem 2.1.** *Let $T > 0$, $\xi \in C^\infty(\mathscr{C};\mathbb{R}) \cap H^2_N$ with $\partial_{111}\xi|_\Gamma = 0$, $(w_0, \theta_0) \in H^2_0 \times H^2_N$, forces $\varphi = \mathbf{\nabla} \wedge \mathbf{\Phi}$ and $\psi$ given by Theorem 1.1, and $\varepsilon > 0$. There exists a small $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ one has*

$$\|S^1_\delta(w_0, \theta_0 - \delta^{-1}\xi, \varphi, \psi) - (w_0 - \partial_1\xi)\|_1 < \varepsilon. \tag{2.3}$$

*Proof.* Thanks to the assumptions on $\xi$, it follows for any $\delta \in (0,1)$ that the pair

$$(w_\delta, \theta_\delta) := S_\delta(w_0, \theta_0 - \delta^{-1}\xi, \varphi, \psi) + (0, \delta^{-1}\xi) \tag{2.4}$$

is well-defined and solves

$$\partial_t w_\delta - \nu\Delta w_\delta + (\mathbf{u}_\delta \cdot \mathbf{\nabla})\, w_\delta = \partial_1(\theta_\delta - \delta^{-1}\xi) + \varphi,$$

$$\partial_t \theta_\delta - \tau\Delta(\theta_\delta - \delta^{-1}\xi) + (\mathbf{u}_\delta \cdot \mathbf{\nabla})(\theta_\delta - \delta^{-1}\xi) = \psi,$$

$$\mathbf{\nabla} \wedge \mathbf{u}_\delta = w_\delta, \quad \mathbf{\nabla} \cdot \mathbf{u}_\delta = 0, \quad \int_\mathscr{C} \mathbf{u}_\delta(\mathbf{x}, t) \cdot \mathbf{g}\, \mathrm{d}\mathbf{x} = 0, \tag{2.5}$$

$$\mathbf{u}_\delta|_\Gamma \cdot \mathbf{n} = 0, \quad w_\delta|_\Gamma = 0, \quad \partial_\mathbf{n}\theta_\delta|_\Gamma = 0,$$

$$w_\delta(\cdot, 0) = w_0, \quad \theta_\delta(\cdot, 0) = \theta_0.$$

In particular, as mentioned in Section 2.1, the temperature $\theta_\delta(\cdot, t)$ is average-free for all $t \in [0, \delta]$; thus, by (2.4), in (2.5) one actually has

$$\int_\mathscr{C} \mathbf{u}_\delta(\mathbf{x}, t) \cdot \mathbf{g}\, \mathrm{d}\mathbf{x} = 0 = \int_0^t \int_\mathscr{C} \theta_\delta(\mathbf{x}, s)\, \mathrm{d}\mathbf{x}\mathrm{d}s, \quad t \in [0, \delta].$$

**Step 1. Ansatz.** To establish $\lim_{\delta \to 0} w_\delta(\cdot, \delta) = (w_0 - \partial_1\xi)$ in $H^1_0$, the following ansatz is made such that $\delta^{-1}\partial_1\xi$, $\tau\delta^{-1}\Delta\xi$, and $\delta^{-1}(\mathbf{u}_\delta \cdot \mathbf{\nabla})\xi$ are canceled in the remainder estimates. More precisely, for each $\delta \in (0, 1)$ we define the remainders

$$q_\delta(\mathbf{x}, t) := w_\delta(\mathbf{x}, t) - w_0(\mathbf{x}) + \delta^{-1}t\partial_1\xi(\mathbf{x}),$$

$$r_\delta(\mathbf{x}, t) := \theta_\delta(\mathbf{x}, t) - \theta_0(\mathbf{x}) + \delta^{-1}t\tau\Delta\xi(\mathbf{x}) - \left(\mathbf{U}_{\delta,\xi}(\mathbf{x}, t) \cdot \mathbf{\nabla}\right)\xi(\mathbf{x}), \tag{2.6}$$

where $\mathbf{U}_{\delta,\xi}$ is uniquely determined as the solution to

$$\mathbf{\nabla} \wedge \mathbf{U}_{\delta,\xi}(\cdot, t) = \delta^{-1}t\left(w_0 - \frac{\delta^{-1}t\partial_1\xi}{2}\right), \quad \mathbf{\nabla} \cdot \mathbf{U}_{\delta,\xi} = 0,$$

$$\mathbf{U}_{\delta,\xi}|_\Gamma \cdot \mathbf{n} = 0, \quad \int_\mathscr{C} \mathbf{U}_{\delta,\xi}(\mathbf{x}, \cdot) \cdot \mathbf{g}\, \mathrm{d}\mathbf{x} = 0. \tag{2.7}$$

6

In view of (2.6), the convergence (2.3) will follow after showing that

$$\lim_{\delta \to 0} \|q_\delta(\cdot, \delta)\|_1 = 0. \tag{2.8}$$

To this end, we consider the problems satisfied by $q_\delta$ and $r_\delta$, which are derived by taking $\partial_t$ in (2.6) and inserting (2.5); namely,

$$\partial_t q_\delta - \nu \Delta q_\delta + (\boldsymbol{Q}_\delta \cdot \boldsymbol{\nabla}) q_\delta = \varphi + \partial_1 (\theta_\delta - r_\delta) + \partial_1 r_\delta - \nu \Delta (q_\delta - w_\delta)$$
$$+ ((\boldsymbol{u}_\delta - \boldsymbol{Q}_\delta) \cdot \boldsymbol{\nabla})(q_\delta - w_\delta) + (\boldsymbol{Q}_\delta \cdot \boldsymbol{\nabla})(q_\delta - w_\delta) + ((\boldsymbol{Q}_\delta - \boldsymbol{u}_\delta) \cdot \boldsymbol{\nabla}) q_\delta \tag{2.9}$$

and

$$\partial_t r_\delta - \tau \Delta r_\delta + (\boldsymbol{Q}_\delta \cdot \boldsymbol{\nabla}) r_\delta = \psi - \tau \Delta (r_\delta - \theta_\delta) + (\boldsymbol{Q}_\delta \cdot \boldsymbol{\nabla})(r_\delta - \theta_\delta)$$
$$+ ((\boldsymbol{Q}_\delta - \boldsymbol{u}_\delta) \cdot \boldsymbol{\nabla}) r_\delta + ((\boldsymbol{Q}_\delta - \boldsymbol{u}_\delta) \cdot \boldsymbol{\nabla})(\theta_\delta - r_\delta) + \delta^{-1}(\boldsymbol{Q}_\delta \cdot \boldsymbol{\nabla}) \xi, \tag{2.10}$$

where $\boldsymbol{Q}_\delta$ is the solution to

$$\boldsymbol{\nabla} \wedge \boldsymbol{Q}_\delta = q_\delta, \quad \boldsymbol{\nabla} \cdot \boldsymbol{Q}_\delta = 0, \quad \boldsymbol{Q}_\delta|_\Gamma \cdot \boldsymbol{n} = 0, \quad \int_{\mathscr{C}} \boldsymbol{Q}_\delta(\boldsymbol{x}, \cdot) \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{x} = 0.$$

By the hypotheses stated in Theorem 2.1, in particular $\partial_1 \xi|_\Gamma = \partial_{111} \xi|_\Gamma = 0$, and due to the definitions in (2.6) and (2.7), the initial and boundary values of the remainders are

$$q_\delta(\cdot, 0) = 0, \quad r_\delta(\cdot, 0) = 0, \quad q_\delta|_\Gamma = 0, \quad \partial_{\boldsymbol{n}} r_\delta|_\Gamma = 0. \tag{2.11}$$

**Step 2. Estimates.** We utilize (1.2), and the Sobolev embeddings $\mathrm{H}^1(\mathscr{C}; \mathbb{R}) \subset \mathrm{L}^4(\mathscr{C}; \mathbb{R})$ and $\mathrm{H}^2(\mathscr{C}; \mathbb{R}) \subset \mathrm{L}^\infty(\mathscr{C}; \mathbb{R})$. First, the equation for $q_\delta$ in (2.9) is multiplied by $q_\delta$, and in a second step by $-\Delta q_\delta$. Then, we integrate over $\mathscr{C} \times (0, t)$ with $t \in [0, \delta]$. Because $\boldsymbol{u}_\delta$ and $\boldsymbol{Q}_\delta$ are divergence free, an also due to the known boundary values of $q_\delta$, $\boldsymbol{Q}_\delta$, and $r_\delta$, integration by parts yields

$$\|q_\delta(\cdot, t)\|^2 + \int_0^t \|q_\delta(\cdot, s)\|_1^2 \, \mathrm{d}s \lesssim \int_0^t \|\varphi(\cdot, s)\|^2 \, \mathrm{d}s + \int_0^t \|(\theta_\delta - r_\delta)(\cdot, s)\|_1^2 \, \mathrm{d}s$$
$$+ \int_0^t \|(q_\delta - w_\delta)(\cdot, s)\|_2^2 \, \mathrm{d}s + \int_0^t \|(q_\delta - w_\delta)(\cdot, s)\|_1^4 \, \mathrm{d}s + \int_0^t \|q_\delta(\cdot, s)\|^4 \, \mathrm{d}s$$
$$+ \int_0^t \left( \|q_\delta(\cdot, s)\|^2 + \|r_\delta(\cdot, s)\|_1^2 \right) \mathrm{d}s =: I_{1,1} + \cdots + I_{1,6}$$

and

$$\|q_\delta(\cdot, t)\|_1^2 + \int_0^t \|\Delta q_\delta(\cdot, s)\|^2 \, \mathrm{d}s$$

$$\lesssim \ell \int_0^t \|q_\delta(\cdot,s)\|_2^2 \, ds + \ell^{-1} \int_0^t \|\varphi(\cdot,s)\|^2 \, ds + \ell^{-1} \int_0^t \|(\theta_\delta - r_\delta)(\cdot,s)\|_1^2 \, ds$$

$$+ \ell^{-1} \int_0^t \|(q_\delta - w_\delta)(\cdot,s)\|_2^2 \, ds + \int_0^t \|(q_\delta - w_\delta)(\cdot,s)\|_2^4 \, ds$$

$$+ \int_0^t \|q_\delta(\cdot,s)\|_1^2 \, ds + \ell^{-1} \int_0^t \|r_\delta(\cdot,s)\|_1^2 \, ds + \int_0^t \|q_\delta(\cdot,s)\|_1^4 \, ds$$

$$=: I_{2,1} + \cdots + I_{2,8}$$

for any $\ell > 0$. Regarding (2.10), similar considerations as above lead to

$$\|r_\delta(\cdot,t)\|^2 + \int_0^t \|r_\delta(\cdot,s)\|_1^2 \, ds$$

$$\lesssim \int_0^t \|\psi(\cdot,s)\|^2 \, ds + \int_0^t \|(r_\delta - \theta_\delta)(\cdot,s)\|_2^2 \, ds + \int_0^t \|(r_\delta - \theta_\delta)(\cdot,s)\|_1^4 \, ds$$

$$+ \int_0^t \|(q_\delta - w_\delta)(\cdot,s)\|_1^4 \, ds + \int_0^t \|r_\delta(\cdot,s)\|^4 \, ds$$

$$+ \delta^{-1} \|\xi\|_3 \int_0^t \left( \|r_\delta(\cdot,s)\|^2 + \|q_\delta(\cdot,s)\|^2 \right) \, ds$$

$$=: J_{1,1} + \cdots + J_{1,6}$$

and

$$\|r_\delta(\cdot,t)\|_1^2 + \int_0^t \|r_\delta(\cdot,s)\|_2^2 \, ds$$

$$\lesssim \ell \int_0^t \|r_\delta(\cdot,s)\|_2^2 \, ds + \ell^{-1} \int_0^t \|\psi(\cdot,s)\|^2 \, ds$$

$$+ \int_0^t \|(r_\delta - \theta_\delta)(\cdot,s)\|_2^4 \, ds + \ell^{-1} \int_0^t \|(r_\delta - \theta_\delta)(\cdot,s)\|_2^2 \, ds$$

$$+ \int_0^t \|(q_\delta - w_\delta)(\cdot,s)\|_2^4 \, ds + \int_0^t \|r_\delta(\cdot,s)\|_1^4 \, ds$$

$$+ \int_0^t \|q_\delta(\cdot,s)\|^4 \, ds + \delta^{-1} \|\xi\|_4 \int_0^t \left( \|r_\delta(\cdot,s)\|_1^2 + \|q_\delta(\cdot,s)\|^2 \right) \, ds$$

$$=: J_{2,1} + \cdots + J_{2,8}.$$

All of the previous estimates are combined, while fixing $\ell > 0$ sufficiently small (independently of $\delta$) such that the terms $I_{2,1}$ and $J_{2,1}$ are absorbed by the resulting left-hand side; *i.e.*,

$$\|q_\delta(\cdot,t)\|_1^2 + \|r_\delta(\cdot,t)\|_1^2 + (1-\ell) \int_0^t \left( \|q_\delta(\cdot,s)\|_2^2 + \|r_\delta(\cdot,s)\|_2^2 \right) \, ds$$

$$\lesssim \sum_{i=1}^6 (I_{1,i} + J_{1,i}) + \sum_{i=2}^8 (I_{2,i} + J_{2,i}), \tag{2.12}$$

where we used that

$$\| \cdot \|_{H^2(\mathscr{C};\mathbb{R})}^2 \lesssim \| \cdot \|_{L^2(\mathscr{C};\mathbb{R})}^2 + \|\Delta \cdot \|_{L^2(\mathscr{C};\mathbb{R})}^2. \tag{2.13}$$

Moreover, for any $a \geq 1$ and $l \in \mathbb{N}$, it follows from (1.2), (2.6), and (2.7) that

$$\|(q_\delta - w_\delta)\|_{L^a((0,\delta);H^l)}^a \lesssim \delta \left(1 + \|w_0\|_l^a + \|\xi\|_{l+1}^a\right),$$

$$\|(r_\delta - \theta_\delta)\|_{L^a((0,\delta);H^l)}^a \lesssim \delta \left(1 + \|w_0\|_{l-1}^{2a} + \|\theta_0\|_l^a + \|\xi\|_{l+3}^{2a}\right),$$

which implies

$$\|(q_\delta - w_\delta)\|_{L^a((0,\delta);H^2)}^a + \|(r_\delta - \theta_\delta)\|_{L^a((0,\delta);H^3)}^a \lesssim \delta. \tag{2.14}$$

Because the forces $(\varphi, \psi)$ are fixed, it follows with the help of (2.14) that

$$\lim_{\delta \to 0} \left( \sum_{i=1}^4 (I_{1,i} + J_{1,i}) + \sum_{i=2}^5 (I_{2,i} + J_{2,i}) \right) = 0.$$

Also, let us emphasize that

$$J_{1,6} + J_{2,8} \lesssim \int_0^\delta \alpha_\delta \left(\|q_\delta(\cdot,s)\|_1^2 + \|r_\delta(\cdot,s)\|_1^2\right) \, ds, \quad \alpha_\delta := \delta^{-1} \max\{1, \|\xi\|_4^2\},$$

where $\int_0^\delta \alpha_\delta \, ds = \max\{1, \|\xi\|_4^2\}$. Now, the remaining integrals $I_{1,i}, I_{2,k}, J_{1,i}$, and $J_{2,k}$, for $i > 4, k > 5$, are good terms for applying Grönwall's inequality in (2.12), which then yields the existence of $c_\delta > 0$ with $\lim_{\delta \to 0} c_\delta = 0$ and such that

$$\|q_\delta(\cdot,t)\|_1^2 + \|r_\delta(\cdot,t)\|_1^2$$
$$\leq \left(c_\delta + C \int_0^t \left(\|q_\delta(\cdot,s)\|_1^4 + \|r_\delta(\cdot,s)\|_1^4\right) \, ds\right) \exp(C \max\{1, \|\xi\|_4^2\})$$

for all $t \in [0, \delta]$. Subsequently, we rename $c_\delta \exp(C \max\{1, \|\xi\|_4^2\})$ again as $c_\delta$ and absorb $\exp(C \max\{1, \|\xi\|_4^2\})$ in the absolute constant $C > 0$. Then, we define

$$f(t) := c_\delta + C \int_0^t \left(\|q_\delta(\cdot,s)\|_1^4 + \|r_\delta(\cdot,s)\|_1^4\right) \, ds.$$

Hence, one has $f'/f^2 \leq C$ and thus $f(\delta) \leq c_\delta(1 - c_\delta \delta C)^{-1} \longrightarrow 0$ as $\delta \longrightarrow 0$, which implies (2.8). $\qquad \square$

## 2.3 Controlling the temperature

We define a constant-in-$x$ vector field whose integral curves all cross the control region $\omega$. This construction is similar to [29, 30] and shall provide a return method type flow (see Remark 2.3) with special structure that facilitates the proof of Theorem 2.6. To begin with, we fix $0 < H_1 < H_2 < 2\pi$ such that

$[-1, 1] \times [H_1, H_2] \subset \omega$, and we further choose a possibly large $K \in \mathbb{N}$ with $l_K := 8\pi/3K < H_2 - H_1/3$. Then, the channel $\mathscr{C}$ is covered by the overlapping rectangles

$$O_i := (-2, 2) \times \left( \frac{3(i-1)l_K}{4}, \frac{3(i-1)l_K}{4} + l_K \right), \quad i \in \{1, \ldots, K\},$$

which are vertical translations of the reference rectangle

$$O := (-2, 2) \times (H_1 + l_K, H_1 + 2l_K) \subset \omega.$$

Moreover, a cutoff function $\chi \in C^\infty(\mathscr{C}; [0, 1])$ with $\mathrm{supp}(\chi) \subset O$, and which only depends on $x_2$, is given via

$$\chi(\boldsymbol{x}) := \widetilde{\chi}(x_2 - H_1 - l_K), \tag{2.15}$$

where $\widetilde{\chi} \in C^\infty(\mathbb{T}; [0, 1])$ is any profile satisfying

$$\mathrm{supp}(\widetilde{\chi}) \subset (0, l_K), \quad \forall x \in (0, l_K/4) \colon \widetilde{\chi}(x) + \widetilde{\chi}(x + 3l_K/4) = 1,$$
$$\widetilde{\chi}(s) = 1 \iff s \in [l_K/4, 3l_K/4].$$

For simplicity, we will occasionally use the abbreviation $\chi(x_2)$ instead of writing $\chi(\boldsymbol{x})$ for $\boldsymbol{x} = [x_1, x_2]^\top \in \mathscr{C}$.

To state the next result, the reference time interval $[0, 1]$ is partitioned equidistantly by

$$0 < t_c^0 < t_a^1 < t_b^1 < t_c^1 < t_a^2 < t_b^2 < t_c^2 < \cdots < t_a^K < t_b^K < t_c^K < 1. \tag{2.16}$$

**Theorem 2.2.** *There exists $\overline{\boldsymbol{y}} = [0, \overline{y}_2]^\top \in C_0^\infty((0, 1); \mathbb{R}^2)$, denoting by $\boldsymbol{\mathcal{Y}}$ its flow obtained via $\frac{\mathrm{d}\boldsymbol{\mathcal{Y}}}{\mathrm{d}t}(\boldsymbol{x}, s, t) = \overline{\boldsymbol{y}}(t)$ and $\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, s, s) = \boldsymbol{x}$, such that one has the properties:*

*P1) Supported in $(t_c^0, t_c^K)$: $\forall t \in [0, t_c^0] \cup [t_c^K, 1] \colon \overline{\boldsymbol{y}}(t) = \boldsymbol{0}$;*

*P2) Closed integral curves: $\forall \boldsymbol{x} \in \mathscr{C} \colon \boldsymbol{\mathcal{Y}}(\boldsymbol{x}, 0, 1) = \boldsymbol{x}$;*

*P3) Stationary visits of $O$: $\forall i \in \{1, \ldots, K\} \colon \boldsymbol{\mathcal{Y}}(O_i, 0, [t_a^i, t_b^i]) = O$.*

*Proof.* The construction from [29, 30] works here as well. Recalling that $\overline{t} := t_c^0$ is the width of the partition in (2.16), one selects $(\beta_i)_{i \in \{1, \ldots, K\}} \subset C_0^\infty((0, \overline{t}); \mathbb{R})$ such that $O_i + \boldsymbol{g} \int_0^{\overline{t}} \beta_i(s) \, \mathrm{d}s = O$. Then, one defines

$$\overline{\boldsymbol{y}}(t) := \begin{cases} \boldsymbol{0} & \text{if } t \in [0, t_c^0] \cup [t_c^K, 1], \\ \boldsymbol{h}_i(t - (3i-2)\overline{t}) & \text{if } t \in (t_c^{i-1}, t_c^i) \text{ for } i \in \{1, \ldots, K\}, \end{cases}$$

where $\boldsymbol{h}_i(t) = \beta_i(t)\boldsymbol{g}$ if $t \in [0, \overline{t}]$, $\boldsymbol{h}_i(t) = \boldsymbol{0}$ if $t \in (\overline{t}, 2\overline{t})$, and $\boldsymbol{h}_i(t) = -\beta_i(t - 2\overline{t})\boldsymbol{g}$ if $t \in [2\overline{t}, 3\overline{t}]$. $\qquad \square$

**Remark 2.3.** Any function $\overline{y}$ of the type provided by Theorem 2.2 is a reference velocity in the return method sense (see [8, Part 2, Chapter 6]). Indeed, it holds $\overline{y}(0) = \overline{y}(1) = 0$, and the integral curves of $\overline{y}$ all cross the control zone. Moreover, a trajectory for a controlled inviscid Boussinesq system is given by $(\overline{u}, \overline{\theta}) :=$ $(\overline{y}, \overline{y}_2' \chi / \int_{\mathscr{C}} \chi(z)\, dz)$, solving in $\mathbb{T}^2 \times [0, 1]$ the controllability problem

$$\partial_t \overline{u} + (\overline{u} \cdot \boldsymbol{\nabla})\overline{u} + \boldsymbol{\nabla}\overline{p} = \overline{\theta}\boldsymbol{\mathcal{g}}, \quad \boldsymbol{\nabla} \cdot \overline{u} = 0, \quad \partial_t \overline{\theta} + (\overline{u} \cdot \boldsymbol{\nabla})\overline{\theta} = \mathbb{I}_\omega \overline{\eta},$$

$$\overline{u}|_\Gamma \cdot \boldsymbol{n} = 0, \quad (\boldsymbol{\nabla} \wedge \overline{u})|_\Gamma = 0, \quad \partial_{\boldsymbol{n}}\overline{\theta}|_\Gamma = 0,$$

$$\overline{u}(\cdot, 0) = \overline{u}(\cdot, 1) = 0, \quad \overline{\theta}(\cdot, 0) = \overline{\theta}(\cdot, 1) = 0,$$

where

$$\overline{p}(\boldsymbol{x}, t) = \int_0^{x_2} \left( \frac{\overline{y}_2'(t)\chi(s)}{\int_{\mathscr{C}} \chi(z)\, dz} - \overline{y}_2'(t) \right) ds, \quad \overline{\eta}(\boldsymbol{x}, t) = \frac{(\overline{y}_2''\chi + \overline{y}_2'(\overline{y} \cdot \boldsymbol{\nabla})\chi)(\boldsymbol{x}, t)}{\int_{\mathscr{C}} \chi(z)\, dz}.$$

Next, we demonstrate the approximate controllability of a linear transport problem with drift $\overline{y}$ and a smooth localized control. Since the drift field $\overline{y}$ is tangential to $\mathscr{C}$ at $\Gamma$, no boundary conditions are required.

**Theorem 2.4.** *Fix any $m \in \mathbb{N}$, $\varepsilon > 0$, and target $\theta_1 \in H^m(\mathscr{C}; \mathbb{R})$. There exists a control $g \in C^\infty(\mathscr{C} \times [0, 1]; \mathbb{R})$ such that the solution $\theta \in C^\infty(\mathscr{C} \times [0, 1]; \mathbb{R})$ to the linear problem*

$$\partial_t \theta + (\overline{y} \cdot \boldsymbol{\nabla})\theta = \mathbb{I}_\omega g, \quad \theta(\cdot, 0) = 0$$

*obeys $\|\theta(\cdot, 1) - \theta_1\|_{m-1} < \varepsilon\|\theta_1\|_m$. Moreover, for a constant $C_\varepsilon > 0$ depending only on $\varepsilon$, the control can be chosen such that $\|g\|_{L^2([0,1]; H^m(\mathscr{C}; \mathbb{R}))} < C_\varepsilon\|\theta_1\|_m$.*

*Proof.* By density and compactness arguments, we take $\widetilde{\theta}_1 \in C^\infty(\mathscr{C}; \mathbb{R})$ with $\|\theta_1 - \widetilde{\theta}_1\|_{m-1} < \varepsilon\|\theta_1\|_m$ and such that $\|\widetilde{\theta}_1\|_{m+1} \leq \widetilde{C}_\varepsilon\|\theta_1\|_m$ for some $\widetilde{C}_\varepsilon > 1$ depending only on $\varepsilon$. Then, we define $\widetilde{\theta}(\cdot, t) := \kappa(t)\widetilde{\theta}_1$, where $\kappa \in C^\infty([0, 1]; [0, 1])$ obeys $\text{supp}(\kappa) \subset (0, 1]$ and $\kappa(1) = 1$. Moreover, we set $\widetilde{g} := \partial_t \widetilde{\theta} + (\overline{y} \cdot \boldsymbol{\nabla})\widetilde{\theta}$ and note that $\widetilde{\theta}$ solves the problem

$$\partial_t \widetilde{\theta} + (\overline{y} \cdot \boldsymbol{\nabla})\widetilde{\theta} = \widetilde{g}, \quad \widetilde{\theta}(\cdot, 0) = 0 \tag{2.17}$$

and satisfies $\|\widetilde{\theta}(\cdot, 1) - \theta_1\|_{m-1} < \varepsilon\|\theta_1\|_m$.

Now, we claim that by using instead of $\widetilde{g}$ the control

$$g(\boldsymbol{x}, t) := \chi(x_2) \sum_{k=1}^K \frac{1}{t_b^k - t_a^k} \mathbb{I}_{[t_a^k, t_b^k]}(t)\widetilde{g}\left( \boldsymbol{\mathcal{Y}}\left(\boldsymbol{x}, t, \frac{t - t_a^k}{t_b^k - t_a^k}\right), \frac{t - t_a^k}{t_b^k - t_a^k} \right), \tag{2.18}$$

the solution to

$$\partial_t \theta + (\overline{y} \cdot \boldsymbol{\nabla})\theta = \mathbb{I}_\omega g, \quad \theta(\cdot, 0) = 0 \tag{2.19}$$

will likewise obey $\|\theta(\cdot, 1) - \theta_1\|_{m-1} < \varepsilon\|\theta_1\|_m$. To see this, we employ the well-known solution formulas for the involved transport problems. First, as $\boldsymbol{\mathcal{Y}}$ is the flow that governs (2.17) and (2.19), it holds

$$\widetilde{\theta}(\boldsymbol{x}, 1) = \int_0^1 \widetilde{g}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, 0, r), r)\, dr, \quad \theta(\boldsymbol{x}, 1) = \int_0^1 g(\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, 0, s), s)\, dr, \tag{2.20}$$

noting that $\text{supp}(\chi) \subset \omega$, and recalling from Theorem 2.2 that $\boldsymbol{\mathcal{Y}}(x, 0, 1) = \boldsymbol{\mathcal{Y}}(x, 1, 0) = x$ for $x \in \mathscr{C}$. Second, it can be shown as follows that both integrals in (2.20) have the same value (see also [29]):

$$
\begin{aligned}
\int_0^1 & g(\boldsymbol{\mathcal{Y}}(x, 0, s), s) \, \mathrm{d}s \\
&= \sum_{k=1}^K \int_0^1 \frac{\mathbb{I}_{[t_a^k, t_b^k]}(s)}{t_b^k - t_a^k} \chi(\boldsymbol{\mathcal{Y}}(x, 0, s)) \widetilde{g}\left( \boldsymbol{\mathcal{Y}}\left(x, 0, \frac{s - t_a^k}{t_b^k - t_a^k}\right), \frac{s - t_a^k}{t_b^k - t_a^k} \right) \mathrm{d}s \\
&= \sum_{k=1}^K \int_0^1 \chi\left( \boldsymbol{\mathcal{Y}}\left(x, 0, r(t_b^k - t_a^k) + t_a^k\right) \right) \widetilde{g}(\boldsymbol{\mathcal{Y}}(x, 0, r), r) \, \mathrm{d}r \qquad (2.21) \\
&= \int_0^1 \widetilde{g}(\boldsymbol{\mathcal{Y}}(x, 0, r), r) \, \mathrm{d}r,
\end{aligned}
$$

where we used *P1-P3* from Theorem 2.2, the substitutions $r = (s - t_a^k)(t_b^k - t_a^k)^{-1}$ for $k \in \{1, \dots, K\}$, and the properties of $\chi$ from (2.15). In particular, regarding the last equality in (2.21), we note that $\chi$ generates the partition of unity $(\chi(\cdot + 3(i-1)l_K/4))_{i \in \{1, \dots, K\}}$, and one can employ *P3*, because $r(t_b^k - t_a^k) + t_a^k \in [t_a^k, t_b^k]$ for all $r \in [0, 1]$; as $\overline{y}$ only depends on time, this yields for any fixed $x \in \mathscr{C}$ that

$$
\sum_{k=1}^K \chi\left( \boldsymbol{\mathcal{Y}}\left(x, 0, r(t_b^k - t_a^k) + t_a^k\right) \right) = 1, \quad r \in [0, 1].
$$

The bound for $\|g\|_{\mathrm{L}^2([0,1];\mathrm{H}^m(\mathscr{C};\mathbb{R}))}$ can be concluded from (2.18) and the definition of $\widetilde{\theta}$, noting that $\overline{y}$ and its flow $\boldsymbol{\mathcal{Y}}$ are smooth and universally fixed. Also, at first it only holds $g \in \mathrm{L}^2([0, 1]; \mathrm{C}^\infty(\mathscr{C}; \mathbb{R}))$, but we can approximate $g$ by a $\mathrm{C}^\infty(\mathscr{C} \times [0, 1]; \mathbb{R})$ version while maintaining $\|\theta(\cdot, 1) - \theta_1\|_{m-1} < \varepsilon \|\theta_1\|_m$. $\qquad \square$

**Remark 2.5.** In the proof of Theorem 2.4 with $m \geq 2$, if one approximates $\theta_1 \in \mathrm{H}_N^m$ in the there-described way by $\widetilde{\theta}_1 \in \mathrm{H}_N^{m+1}$, then a direct calculation, using the properties of $\overline{y}$ and $\chi$ in (2.18), provides $\partial_n \mathbb{I}_\omega g(\cdot, t)|_\Gamma = 0$ for $t \in [0, 1]$.

Now, we steer the temperature in the nonlinear problem, while ensuring that the final vorticity is a small perturbation of the initial one; for the torus case, see [29]. The employed return method argument is inspired by [7] (see also [8]) and follows the implementations in [17, 28, 29]. The idea is to view a scaled solution to (1.1) on a small time interval as a perturbation of an accordingly scaled return method trajectory from Remark 2.3.

**Theorem 2.6.** *Let $T > 0$, $\varepsilon > 0$, $(w_0, \theta_0, \theta_1) \in \mathrm{H}_0^2 \times \mathrm{H}_N^3 \times \mathrm{H}_N^3$, and forces $\varphi = \boldsymbol{\nabla} \wedge \boldsymbol{\Phi}$ and $\psi$ given by Theorem 1.1. There exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$ there is a control $\eta \in \mathrm{C}^\infty(\mathscr{C} \times [0, \delta]; \mathbb{R})$ with*

$$
\|S_\delta(w_0, \theta_0, \varphi, \psi + \eta) - (w_0, \theta_1)\|_{\mathrm{H}^1 \times \mathrm{H}^2} < \varepsilon. \qquad (2.22)
$$

*Moreover, the velocity $\boldsymbol{u}_\delta$, associated to $w_0$, $\theta_0$, $\varphi$, $\psi$, and $\eta$ via (2.1), obeys*

$$\int_{\mathscr{C}} \boldsymbol{u}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{x} = [0, \overline{y}_{2,\delta}(t)]^\top = \overline{\boldsymbol{y}}_\delta(t) := \delta^{-1} \overline{\boldsymbol{y}}(\delta^{-1} t), \quad t \in [0, \delta].$$

*Proof.* First, a family of auxiliary controls is constructed via Theorem 2.4. Second, a controlled nonlinear trajectory is fixed. Third, asymptotic expansions are proposed for verifying (2.22). Finally, the remainder estimates are given.

**Step 1. Controlling a family of linear problems.** For $\delta \in (0,1)$, we denote by $\widetilde{\Theta}_\delta$ the unique solution to the uncontrolled transport equation with scaled temperature initial data

$$\partial_t \widetilde{\Theta}_\delta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \widetilde{\Theta}_\delta = 0, \quad \widetilde{\Theta}_\delta(\cdot, 0) = \delta\theta_0, \tag{2.23}$$

emphasizing that $\partial_{\boldsymbol{n}} \widetilde{\Theta}_\delta(\cdot, t)|_\Gamma = 0$ for all $t \in [0,1]$ due to $\theta_0 \in \mathrm{H}_N^3$. Since $\boldsymbol{\mathcal{Y}}$ is the flow of $\overline{\boldsymbol{y}}$ from Theorem 2.2, we have $\widetilde{\Theta}_\delta(\cdot, 1) = \delta\theta_0$. Now, for any $\delta \in (0,1)$, we apply Theorem 2.4, in the way described by Remark 2.5, with the target state $\delta(\theta_1 - \theta_0)$. This yields a family of smooth controls $(g_\delta)_{\delta \in (0,1)}$, spatially supported in $\omega$, such that the solution to the problem

$$\partial_t \widehat{\Theta}_\delta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \widehat{\Theta}_\delta = \mathbb{I}_\omega g_\delta, \quad \widehat{\Theta}_\delta(\cdot, 0) = 0 \tag{2.24}$$

satisfies $\|\widehat{\Theta}_\delta(\cdot, 1) - \delta(\theta_1 - \theta_0)\|_2 < \varepsilon\delta\|\theta_1 - \theta_0\|_3$ and $\partial_{\boldsymbol{n}} \widehat{\Theta}_\delta(\cdot, t)|_\Gamma = 0$. Using the linearity of (2.23) and (2.24), it follows that $\overline{\vartheta}_\delta := \widetilde{\Theta}_\delta + \widehat{\Theta}_\delta$ satisfies

$$\partial_t \overline{\vartheta}_\delta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \overline{\vartheta}_\delta = \mathbb{I}_\omega g_\delta,$$
$$\overline{\vartheta}_\delta(\cdot, 0) = \delta\theta_0, \quad \|\overline{\vartheta}_\delta(\cdot, 1) - \delta\theta_1\|_2 < \varepsilon\delta\|\theta_1 - \theta_0\|_3. \tag{2.25}$$

However, in the steps below, we will work with the following average-free version:

$$\widetilde{\vartheta}_\delta(\boldsymbol{x}, t) := \overline{\vartheta}_\delta(\boldsymbol{x}, t) - \frac{\chi(x_2) \int_0^t \int_{\mathscr{C}} g_\delta(\boldsymbol{z}, s) \, \mathrm{d}\boldsymbol{z}\mathrm{d}s}{\int_{\mathscr{C}} \chi(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z}},$$

which satisfies (due to Remark 2.5, (2.25), and $\theta_0, \theta_1 \in \mathrm{H}_N^3$) the initial, target, and boundary conditions

$$\widetilde{\vartheta}_\delta(0) = \delta\theta_0, \quad \|\widetilde{\vartheta}_\delta(1) - \delta\theta_1\|_2 < \varepsilon\delta\|\theta_1 - \theta_0\|_3, \quad \partial_{\boldsymbol{n}} \widetilde{\vartheta}_\delta|_\Gamma = 0. \tag{2.26}$$

Moreover, together with the control

$$\widetilde{\eta}_\delta(\boldsymbol{x}, t) := g_\delta(\boldsymbol{x}, t) - \frac{\overline{y}_2 \chi'(x_2) \int_0^t \int_{\mathscr{C}} g_\delta(\boldsymbol{z}, s) \, \mathrm{d}\boldsymbol{z}\mathrm{d}s + \chi(x_2) \int_{\mathscr{C}} g_\delta(\boldsymbol{z}, t) \, \mathrm{d}\boldsymbol{z}}{\int_{\mathscr{C}} \chi(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z}},$$
$$\tag{2.27}$$

it holds

$$\partial_t \widetilde{\vartheta}_\delta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \widetilde{\vartheta}_\delta = \mathbb{I}_\omega \widetilde{\eta}_\delta. \tag{2.28}$$

13

Meanwhile, we define the corresponding function $\widetilde{v}_\delta$ as the solution to

$$\partial_t \widetilde{v}_\delta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\widetilde{v}_\delta = \partial_1 \widetilde{\vartheta}_\delta, \quad \widetilde{v}_\delta(\cdot, 0) = w_0, \tag{2.29}$$

observing that $\widetilde{v}_\delta|_\Gamma = 0$ due to (2.26) and the choice of $w_0$. Direct estimates based on the solution representations for the transport problems in (2.23)–(2.25), (2.28), and (2.29) provide

$$\sup_{t \in [0,1]} \|\widetilde{v}_\delta(\cdot, t) - w_0\|_2 + \sup_{t \in [0,1]} \|\widetilde{\vartheta}_\delta(\cdot, t)\|_3 \lesssim C_\varepsilon \delta, \tag{2.30}$$

where $C_\varepsilon > 1$ is the fixed constant from Theorem 2.4 depending only on $\varepsilon$.

**Step 2. Controls for the nonlinear problem.** Given $(\widetilde{\eta}_\delta)_{\delta \in (0,1)}$ from (2.27), we aim to relate, for small $\delta$, the final states of suitably controlled trajectories of the nonlinear problem (2.1) to those of the linear systems with parameter $\delta$ constituted by (2.28) and (2.29). Hereto, in the next step (see (2.34)), we will view $\boldsymbol{u}_\delta$ on the short time interval $[0, \delta]$ as a perturbation of the leading order profile $\overline{\boldsymbol{y}}_\delta(t) = \delta^{-1} \overline{\boldsymbol{y}}(\delta^{-1} t)$. But for this to be feasible, the velocity $\boldsymbol{u}_\delta$ should have the vertical average $\int_{\mathscr{C}} \boldsymbol{u}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{g}(\boldsymbol{x}) \, d\boldsymbol{x} = \overline{\boldsymbol{y}}_\delta(t)$ for all $t \in [0, \delta]$. To ensure this property, we can act on the velocity average through the temperature control. Namely, for each $\delta \in (0, 1)$, we denote (see Remark 2.7)

$$\eta_\delta := \delta^{-2} \widetilde{\eta}_\delta(\cdot, \delta^{-1} \cdot) + \frac{\overline{y}_{2,\delta}'' \chi - \tau \overline{y}_{2,\delta}' \Delta\chi + \overline{y}_{2,\delta}'(\widetilde{\boldsymbol{u}}_\delta \cdot \boldsymbol{\nabla})\chi}{\int_{\mathscr{C}} \chi(x_2) \, d\boldsymbol{x}}, \tag{2.31}$$

where $\chi$ is from (2.15) and $\widetilde{\boldsymbol{u}}_\delta$ is the solution to

$$\boldsymbol{\nabla} \wedge \widetilde{\boldsymbol{u}}_\delta(\cdot, t) = S_t^1(w_0, \theta_0, \varphi, \psi + \delta^{-2}\widetilde{\eta}_\delta(\cdot, \delta^{-1}\cdot) + \overline{y}_{2,\delta}''),$$

$$\boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{u}}_\delta = 0, \quad \widetilde{\boldsymbol{u}}_\delta|_\Gamma \cdot \boldsymbol{n} = 0, \quad \int_{\mathscr{C}} \widetilde{\boldsymbol{u}}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{g} \, d\boldsymbol{x} = \overline{\boldsymbol{y}}_\delta(t),$$

Next, we fix the controlled trajectory

$$(w_\delta, \theta_\delta)(\cdot, t) := S_t(w_0, \theta_0, \varphi, \psi + \eta_\delta), \quad t \in [0, \delta] \tag{2.32}$$

and denote by $\boldsymbol{u}_\delta$ the associated velocity. Now, due to the above constructions, we can verify that

$$\boldsymbol{\nabla} \wedge \boldsymbol{u}_\delta = w_\delta, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u}_\delta = 0, \quad \boldsymbol{u}_\delta|_\Gamma \cdot \boldsymbol{n} = 0, \quad \int_{\mathscr{C}} \boldsymbol{u}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{g} \, d\boldsymbol{x} = \overline{\boldsymbol{y}}_\delta(t)$$

and

$$\boldsymbol{u}_\delta = \widetilde{\boldsymbol{u}}_\delta. \tag{2.33}$$

14

Indeed, each $\widetilde{\eta}_\delta(\cdot, t)$ is average-free for almost all $t \in (0, 1)$ because of (2.27), and the second term in the right-hand sides of (2.31) ensures that (see (2.1) and (2.2))

$$\theta_\delta(\cdot, t) = S_t^2(w_0, \theta_0, \varphi, \psi + \delta^{-2}\widetilde{\eta}_\delta(\cdot, \delta^{-1}\cdot)) + \frac{\overline{y}_{2,\delta}' \chi}{\int_\mathscr{C} \chi(x_2)\, \mathrm{d}x},$$

$$\int_\mathscr{C} \boldsymbol{u}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{g}\, \mathrm{d}x = \int_0^t \int_\mathscr{C} \theta_\delta(\boldsymbol{x}, s)\, \mathrm{d}x\mathrm{d}s = \overline{y}_{2,\delta}(t).$$

In order to confirm (2.33), one notes that $\overline{\boldsymbol{y}}_\delta'(t) - \overline{\boldsymbol{y}}_\delta'(t)\chi / \int_\mathscr{C} \chi(z)\, \mathrm{d}z$ is curl-free and average-free for $t \in [0, \delta]$, implying $\overline{\boldsymbol{y}}_\delta' - \overline{\boldsymbol{y}}_\delta'\chi / \int_\mathscr{C} \chi(z)\, \mathrm{d}z = \nabla q$ with smooth $q$. Hence, $\boldsymbol{u}_\delta$ and $\widetilde{\boldsymbol{u}}_\delta$ solve the same velocity equation with identical data. See also Remark 2.3.

**Remark 2.7.** The definition in (2.31) has the effect, that if $\rho$ solves $\partial_t \rho - \tau \Delta \rho + (\widetilde{\boldsymbol{u}}_\delta \cdot \nabla)\rho = \delta^{-2}\widetilde{\eta}_\delta(\cdot, \delta^{-1}\cdot)$, then $\widetilde{\rho} := \rho + \chi \overline{y}_{2,\delta}'$ obeys $\partial_t \widetilde{\rho} - \tau \Delta \widetilde{\rho} + (\widetilde{\boldsymbol{u}}_\delta \cdot \nabla)\widetilde{\rho} = \eta_\delta$.

**Step 3. Asymptotic expansions.** Given the trajectory $(w_\delta, \theta_\delta)$ defined in (2.32), we make on the time interval $[0, \delta]$ an ansatz of the form

$$w_\delta = z_\delta + q_\delta, \quad \boldsymbol{u}_\delta = \overline{\boldsymbol{y}}_\delta + \boldsymbol{Z}_\delta + \boldsymbol{Q}_\delta, \quad \theta_\delta = \vartheta_\delta + \frac{\overline{y}_{2,\delta}' \chi}{\int_\mathscr{C} \chi(x_2)\, \mathrm{d}x} + r_\delta, \quad (2.34)$$

where

$$z_\delta(\cdot, t) := \widetilde{v}_\delta(\cdot, \delta^{-1}t), \quad \vartheta_\delta(\cdot, t) := \delta^{-1}\widetilde{\vartheta}_\delta(\cdot, \delta^{-1}t)$$

are determined through (2.28) and (2.29), while the vector fields $\boldsymbol{Z}_\delta$ and $\boldsymbol{Q}_\delta$ are the unique functions satisfying

$$\nabla \wedge \boldsymbol{Z}_\delta = z_\delta, \quad \nabla \cdot \boldsymbol{Z}_\delta = 0, \quad \boldsymbol{Z}_\delta|_\Gamma \cdot \boldsymbol{n} = 0, \quad \int_\mathscr{C} \boldsymbol{Z}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{g}\, \mathrm{d}x = 0,$$

$$\nabla \wedge \boldsymbol{Q}_\delta = q_\delta, \quad \nabla \cdot \boldsymbol{Q}_\delta = 0, \quad \boldsymbol{Q}_\delta|_\Gamma \cdot \boldsymbol{n} = 0, \quad \int_\mathscr{C} \boldsymbol{Q}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{g}\, \mathrm{d}x = 0.$$

It should be emphasized that, due to the boundary data of $\theta_0$ and $\theta_1$, and the Dirichlet condition for $w_0$, one has $z_\delta|_\Gamma = 0$ and $\partial_{\boldsymbol{n}} \vartheta_\delta|_\Gamma = 0$; see also Remark 2.5, (2.26), (2.28), and (2.29). In view of (2.26), (2.34), and the compact support of $\overline{\boldsymbol{y}}$, it remains to show

$$\|q_\delta(\cdot, \delta)\|_1 + \|r_\delta(\cdot, \delta)\|_2 \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0. \quad (2.35)$$

The limit (2.35) can be verified via energy estimates for the equations satisfied by $q_\delta$ and $r_\delta$, namely

$$\partial_t q_\delta - \nu \Delta q_\delta + \left((\overline{\boldsymbol{y}}_\delta + \boldsymbol{Z}_\delta + \boldsymbol{Q}_\delta) \cdot \nabla\right) q_\delta + (\boldsymbol{Q}_\delta \cdot \nabla)z_\delta = F_\delta + \partial_1 r_\delta,$$
$$\partial_t r_\delta - \tau \Delta r_\delta + \left((\overline{\boldsymbol{y}}_\delta + \boldsymbol{Z}_\delta + \boldsymbol{Q}_\delta) \cdot \nabla\right) r_\delta + (\boldsymbol{Q}_\delta \cdot \nabla)\vartheta_\delta = G_\delta, \quad (2.36)$$
$$q_\delta(\cdot, 0) = 0, \quad r_\delta(\cdot, 0) = 0, \quad q_\delta|_\Gamma = 0, \quad \partial_{\boldsymbol{n}} r_\delta|_\Gamma = 0,$$

where $F_\delta := \varphi - (\boldsymbol{Z}_\delta \cdot \nabla)z_\delta + \nu \Delta z_\delta$ and $G_\delta := \psi - (\boldsymbol{Z}_\delta \cdot \nabla)\vartheta_\delta + \tau \Delta \vartheta_\delta$. To obtain (2.36), one inserts (2.32) and (2.34) into (2.1), and further utilizes that $\partial_1 \chi \equiv 0$ by (2.15), as well as the expression of $\eta_\delta$ from (2.31) in conjunction with (2.33).

15

**Step 4. Estimates.** Let $t \in [0, \delta]$, and recall the Sobolev embeddings $\mathrm{H}^1(\mathscr{C}; \mathbb{R}) \subset \mathrm{L}^4(\mathscr{C}; \mathbb{R})$ and $\mathrm{H}^2(\mathscr{C}; \mathbb{R}) \subset \mathrm{L}^\infty(\mathscr{C}; \mathbb{R})$. We multiply in (2.36) respectively with $(-\Delta)^i q_\delta$ and $(-\Delta)^i r_\delta$ for $i = 0, 1$, and then use integration by parts. By employing the boundary values of the involved functions, (1.2), and (2.13), we find

$$
\|q_\delta(\cdot, t)\|_1^2 + \|r_\delta(\cdot, t)\|_1^2 + (1 - \ell) \int_0^t \left( \|q_\delta(\cdot, s)\|_2^2 + \|r_\delta(\cdot, s)\|_2^2 \right) \mathrm{d}s
$$

$$
\lesssim \int_0^t \ell^{-1} \left( \|F_\delta(\cdot, s)\|^2 + \|G_\delta(\cdot, s)\|^2 + \|q_\delta(\cdot, s)\|_1^2 + \|r_\delta(\cdot, s)\|_1^2 \right) \mathrm{d}s
$$

$$
+ \int_0^t \ell^{-1} \left( \|q_\delta(\cdot, s)\|_1^4 + \|r_\delta(\cdot, s)\|_1^4 + \|\vartheta_\delta(\cdot, s)\|_1^4 \right) \mathrm{d}s \tag{2.37}
$$

$$
+ \int_0^t \left( \|z_\delta(\cdot, s)\|_2 + |\overline{\boldsymbol{y}}_\delta(\cdot, s)| \right) \left( \|q_\delta(\cdot, s)\|_1^2 + \|r_\delta(\cdot, s)\|_1^2 \right) \mathrm{d}s,
$$

where $\ell \in (0, 1)$ is fixed independently of $\delta$. Next, after acting in the equation for $r_\delta$ in (2.36) with $\partial_i$, where $i \in \{1, 2\}$, it follows that

$$
\partial_t \partial_i r_\delta - \tau \Delta \partial_i r_\delta + \left( (\overline{\boldsymbol{y}}_\delta + \boldsymbol{Z}_\delta + \boldsymbol{Q}_\delta) \cdot \boldsymbol{\nabla} \right) \partial_i r_\delta + (\boldsymbol{Q}_\delta \cdot \boldsymbol{\nabla}) \partial_i \vartheta_\delta
$$
$$
= \partial_i G_\delta - \left( (\partial_i \boldsymbol{Z}_\delta + \partial_i \boldsymbol{Q}_\delta) \cdot \boldsymbol{\nabla} \right) r_\delta - (\partial_i \boldsymbol{Q}_\delta \cdot \boldsymbol{\nabla}) \vartheta_\delta.
$$

Multiplying with $-\Delta \partial_i r_\delta$, integrating by parts, and using the known boundary values of the involved functions, we obtain together with (2.37) that

$$
\|q_\delta(\cdot, t)\|_1^2 + \|r_\delta(\cdot, t)\|_2^2 + (1 - \ell) \int_0^t \left( \|q_\delta(\cdot, s)\|_2^2 + \|r_\delta(\cdot, s)\|_3^2 \right) \mathrm{d}s
$$

$$
\lesssim \int_0^t \ell^{-1} \left( \|F_\delta(\cdot, s)\|^2 + \|G_\delta(\cdot, s)\|_1^2 + \|q_\delta(\cdot, s)\|_1^2 + \|r_\delta(\cdot, s)\|_2^2 \right) \mathrm{d}s
$$

$$
+ \int_0^t \ell^{-1} \left( \|q_\delta(\cdot, s)\|_1^4 + \|r_\delta(\cdot, s)\|_2^4 + \|\vartheta_\delta(\cdot, s)\|_2^4 \right) \mathrm{d}s
$$

$$
+ \int_0^t \left( \|z_\delta(\cdot, s)\|_2 + |\overline{\boldsymbol{y}}_\delta(\cdot, s)| \right) \left( \|q_\delta(\cdot, s)\|_1^2 + \|r_\delta(\cdot, s)\|_2^2 \right) \mathrm{d}s,
$$

with fixed $\ell \in (0, 1)$ independent of $\delta$. Moreover, the forces $\varphi = \boldsymbol{\nabla} \wedge \boldsymbol{\Phi}$ and $\psi$ are fixed in Theorem 1.1, and the norms $\|z_\delta(\cdot, t)\|_2$ and $\|\vartheta_\delta(\cdot, t)\|_3$ are for $t \in [0, \delta]$ bounded by an absolute constant. Thus, by noting that

$$
\|(\boldsymbol{Z}_\delta \cdot \boldsymbol{\nabla}) z_\delta(\cdot, t)\| \leq \|z_\delta(\cdot, t)\|_1^2, \quad \|\Delta z_\delta(\cdot, t)\| \leq \|z_\delta(\cdot, t)\|_2,
$$

$$
\|(\boldsymbol{Z}_\delta \cdot \boldsymbol{\nabla}) \vartheta_\delta(\cdot, t)\|_1 \leq \|z_\delta(\cdot, t)\|_1^2 + \|\vartheta_\delta(\cdot, t)\|_2^2, \quad \|\Delta \vartheta_\delta(\cdot, t)\|_1 \leq \|\vartheta_\delta(\cdot, t)\|_3,
$$

we can deduce (using the substitution $s = \delta\sigma$)

$$
\lim_{\delta \to 0} \int_0^\delta \left( \|F_\delta(\cdot, s)\| + \|G_\delta(\cdot, s)\|_1 \right) \mathrm{d}s = 0,
$$

$$
\int_0^t \left( \|z_\delta(\cdot, s)\|_2 + |\overline{\boldsymbol{y}}_\delta(\cdot, s)| \right) \mathrm{d}s = \int_0^1 \left( \delta \|\widetilde{v}_\delta(\cdot, \sigma)\|_2 + |\overline{\boldsymbol{y}}(\sigma)| \right) \mathrm{d}\sigma \leq M_\delta,
$$

16

where $(M_\delta)_{\delta \in (0,1)}$ is a bounded family. In particular, thanks to (2.30), one has $\lim_{\delta \to 0} M_\delta = \sup_{t \in [0,1]} |\bar{y}(t)|$. In conclusion, using Grönwall's inequality,

$$\|q_\delta(\cdot, t)\|_1^2 + \|r_\delta(\cdot, t)\|_2^2 \lesssim c_\delta + \int_0^t \left\| \left( q_\delta(\cdot, s)\|_1^4 + \|r_\delta(\cdot, s)\|_2^4 \right) \right. ds,$$

where $\lim_{\delta \to 0} c_\delta = 0$. Like in the proof of Theorem 2.1, comparing with $f'/f^2 \leq C$ for an absolute constant $C > 0$, the limit (2.35) follows. $\qquad\square$

## 2.4 Conclusion

To conclude Theorem 1.1, it suffices, in view of the estimate (1.2), to obtain for any $\varepsilon > 0$ a control $\eta \in C^\infty(\mathscr{C} \times [0, T])$ with $\mathrm{supp}(\eta) \subset \omega \times (0, T)$ such that the corresponding solution to (1.1) obeys

$$\|(\boldsymbol{\nabla} \wedge \boldsymbol{u})(\cdot, T) - w_T\|_1 + \|\theta(\cdot, T) - \theta_T\|_2 + |\int_{\mathscr{C}} \boldsymbol{u}(\boldsymbol{x}, T) \cdot \boldsymbol{g} \, d\boldsymbol{x}| < \varepsilon,$$

where $w_T := \boldsymbol{\nabla} \wedge \boldsymbol{u}_T$.

In what follows, we specify a control $\eta$, and the corresponding controlled trajectory, on respective time intervals $[0, T_1]$, $[T_1, T_2]$, $[T_2, T_3]$, and $[T_3, T]$, with $0 < T_1 < T_2 < T_3 < T$.

**Step 1. Smoothing.** Take $T_1 \in (0, T)$ such that the corresponding solution $(w, \theta)$ to (2.1) with "zero control", *i.e.*, $\eta = 0$, satisfies

$$(w_{T_1}, \theta_{T_1}) := (w(\cdot, T_1), \theta(\cdot, T_1)) \in H_0^2 \times H_N^3.$$

Following the literature mentioned in Remarks 1.2 and 1.3, the assumptions on $(\boldsymbol{\Phi}, \psi)$ in Theorem 1.1 facilitate parabolic smoothing of $(\boldsymbol{u}, \theta)$ up to the regularity

$$L^\infty((t_0, T); H^2(\mathscr{C}; \mathbb{R}^{2+1})) \cap L^2((0, T); H^3(\mathscr{C}; \mathbb{R}^{2+1})),$$

for any $t_0 > 0$, with the details being provided, for instance, by [5, Lemma 2.1] without external forces; if one would consider smooth external forces, for example, $(\boldsymbol{\Phi}, \psi) = (\boldsymbol{0}, 0)$, one would obtain solutions that are smooth for $t > 0$. See also [32, Remark 3.2] for such an argument (written for the no-slip boundary condition) taking into account external forces.

Owing to the basic global wellposedness and regularity properties of (1.1), we take $T_1 \in (0, T)$ so close to $T$ that every uncontrolled trajectory of (2.1) starting at $t = \widetilde{T}$ inside the $2\varepsilon/3$-ball with center $(w_T, \theta_T)$ in $H^1(\mathscr{C}; \mathbb{R}) \times H^2(\mathscr{C}; \mathbb{R})$ will not cross the boundary of the $\varepsilon$-ball with center $(w_T, \theta_T)$ in $H^1(\mathscr{C}; \mathbb{R}) \times H^2(\mathscr{C}; \mathbb{R})$ until $t = \widetilde{T} + (T - T_1)$.

**Step 2. Reaching a special temperature state.** Let $\xi \in C^\infty(\mathscr{C}; \mathbb{R})$ be any average-free profile with $\partial_1 \xi|_\Gamma = \partial_{111} \xi|_\Gamma = 0$ and so that

$$\|(w_{T_1} - \partial_1 \xi) - w_T\|_1 < \varepsilon/3.$$

Such a choice is possible due to the boundary conditions satisfied by $w_{T_1} = w(\cdot, T_1)$ and $w_T$, and by the density of $C_0^\infty(\mathscr{C}; \mathbb{R})$ in $H_0^1$, where $C_0^\infty(\mathscr{C}; \mathbb{R})$ denotes the smooth functions with compact support in $\mathscr{C}$. Indeed, by density, we first take $h \in C_0^\infty(\mathscr{C}; \mathbb{R})$ with $\|(w_{T_1} - h) - w_T\|_1 < \varepsilon/3$. Next, we integrate $h$ in the $x_1$-direction and subsequently correct the average; namely

$$\widehat{\xi}(x_1, x_2) := \int_{-1}^{x_1} h(s, x_2) \, \mathrm{d}s, \quad \xi := \widehat{\xi} - \int_\mathscr{C} \widehat{\xi}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x},$$

emphasizing that $\partial_1 \xi|_\Gamma = \partial_{111} \xi|_\Gamma = 0$, and that $\xi$ has zero average. Now, using Theorem 2.6 and the smoothing property, we fix a small $\widetilde{\delta} \in (0, T - T_1)$ and a control $\eta \in C^\infty(\mathscr{C} \times [0, T])$ such that the corresponding solution $(\boldsymbol{u}, w, \theta)$ to (2.1) starting from $t = T_1$ with data $(w_{T_1}, \theta_{T_1})$ satisfies at the time $T_2 := T_1 + \widetilde{\delta}$ the conditions

$$\|w_{T_2} - w_{T_1}\|_1 < \beta, \quad \|\theta_{T_2} - \gamma^{-1}\xi\|_2 < \kappa, \quad \int_\mathscr{C} \boldsymbol{u}_{T_2} \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{x} = 0,$$

$$(\boldsymbol{u}_{T_2}, w_{T_2}, \theta_{T_2}) := (\boldsymbol{u}, w, \theta)(\cdot, T_2)) \in H_0^2 \times H_N^3,$$

where $\beta > 0$, $\gamma > 0$, and $\kappa > 0$ are selected so small that one has (2.3) with $\varepsilon/2$ when applying Theorem 2.1 with initial data $(w_{T_2}, \theta_{T_2})$ and $\delta = \gamma$ and $T_2 + \gamma < T$.

**Step 3. Controlling the vorticity.** Theorem 2.1 provides a time $T_3 \in (T_2, T)$, for instance, $T_3 = T_2 + \gamma$, such that without using any control the solution $(\boldsymbol{u}, w, \theta)$ to (2.1) on the interval $[T_2, T_3]$ issued at $t = T_2$ from $(w_{T_2}, \theta_{T_2})$, as obtained in the previous step, satisfies

$$\|w(\cdot, T_3) - w_T\|_1 < \varepsilon/2, \quad \int_\mathscr{C} \boldsymbol{u}(\boldsymbol{x}, T_3) \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{x} = 0.$$

**Step 4. Correcting the temperature.** A final application of Theorem 2.6 provides a time $T_4 \in (T_3, T)$ such that the solution $(w, \theta)$ to (1.1) on the interval $[T_3, T_4]$ with initial data $w(\cdot, T_3)$ and $\theta(\cdot, T_3)$, as fixed above, obeys

$$\|w(\cdot, T_4) - w_T\|_1 + \|\theta(\cdot, T_4) - \theta_T\|_2 < 2\varepsilon/3.$$

As $T_1$ was at the beginning chosen sufficiently close to $T$, the argument is complete.

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